Lech's inequality and stability of local rings

(j.w. Ilya Smirnov)

<u>Theorem (Lech)</u> (R,m) = local $I \subseteq R$ m-primary ideal $d=\dim(R)$

60's!

Then $(*)$ $e(I) \leq d!\, e(R)\, \ell(R/I)$

Here, $e(I) = \lim_{n\to\infty} \frac{\ell(R/I^n)}{n^d} \cdot d! \in \mathbb{Z}_{>0}$ $\quad e(R) := e(m)$

<u>Remarks</u> ① $I = J^N$ then $e(J^N) = N^d e(J)$

$\ell(R/J^N) \sim N^d e(J) \cdot \frac{1}{d!}$

$\Rightarrow e(J^N) \sim d!\, \ell(R/J^N)$

$\Rightarrow$ $(*)$ is asymptotically sharp if $e(R)=1$

(e.g. R = regular)

② $R = k[\![x_1, \cdots, x_d]\!]$ I = m-primary monomial ideal.

$NP(I)$ = convex hull of $\{\underline{a} \in \mathbb{R}^d_{\geq 0} \mid \underline{x}^{\underline{a}} \in I\}$

$e(I) = d!\, \mathrm{Vol}(\mathbb{R}^d_{\geq 0} \setminus NP(I))$

$\ell(R/I)$ = # of integer points in $\mathbb{R}^d_{\geq 0} \setminus NP(I)$

<u>Definition</u> (R, m) = local ring $C_{LM}(R) := \sup\limits_{\sqrt{I}=m} \left\{ \frac{e(I)}{d!\, \ell(R/I)} \right\}$

is called <u>Lech–Mumford constant</u> of R

<u>Rmk</u>: ① $C_{LM}(R) = C_{LM}(\hat{R})$ can restrict to $I = \bar{I}$ in sup

② $\underbrace{1 \leq C_{LM}(R)}_{\text{take } I = \mathfrak{J}^N} \underbrace{\leq e(R)}_{\text{Lech's inequality } (*)}$

<u>Theorem</u> (Huneke–M–Quy–Smirnov, MS) $C_{LM}(R) = e(R)$ if and only if one of the following holds

① $\dim(R) \leq 1$

② $\dim(R) \geq 2$ and $e(\hat{R}_{red}) = 1$

<u>Theorem</u> (Mumford) X = Projective / k = field $\mathcal{L}$ = ample

Suppose $(X, \mathcal{L})$ is <u>asymptotically Chow semistable</u>: i.e. $\forall n \gg 0$ the Chow form $X \overset{|\mathcal{L}^n|}{\hookrightarrow} \mathbb{P}^N$ is semistable under the natural SL_{N+1}-action. Then $\forall x \in X$ closed point, we have

$$C_{LM}(\underbrace{\mathcal{O}_{X,x}}_{\text{local ring}}[\![t]\!]) = 1.$$

<u>Lemma (Mumford)</u>

$$e(R) \geq C_{LM}(R) \geq C_{LM}(R[\![t]\!]) \geq C_{LM}(R[\![t_1, t_2]\!]) \cdots\cdots \geq 1$$

Definition A local ring (R,m) is called

① **semistable** if $C_{LM}(R[\![t]\!]) = 1$
①' $\exists$ a notion of **stable** (omitted)
] Mumford

② **Lech-stable** if $C_{LM}(R) = 1$
③ **lim-stable** if $\lim_{n\to\infty} C_{LM}(R[\![t_1, \dots, t_n]\!]) = 1$
] MS

Observations:

$$\text{Lech-stable} \Longrightarrow \text{semistable} \Longrightarrow \text{lim-stable}$$

$$\text{Lech-stable} \dashrightarrow \text{stable} \dashrightarrow \text{semistable}$$

Exercise: $\boxed{\dim(R) = 0}$ then $C_{LM}(R[\![t_1, \dots, t_n]\!]) = \ell(R)$ $\forall n$

In particular, Lech-stable $\Leftrightarrow$ semistable $\Leftrightarrow$ lim-stable
($\Leftrightarrow R =$ field)

Example: $R = \dfrac{k[\![x,y]\!]}{xy}$ $\quad\dim(R) = 1$ $\quad C_{LM}(R) = e(R) = 2$
$R \neq$ Lech-stable

Mumford: R is semistable.
i.e., $C_{LM}\left(\dfrac{k[\![x,y,z]\!]}{xy}\right) = 1$

(MS)
Theorem $\boxed{\dim(R) = 1}$ $\quad R =$ complete. CM

① R is Lech-stable $\Leftrightarrow R =$ DVR
② R is semistable $\Leftrightarrow$ lim-stable $\Leftrightarrow$ either $R =$ DVR or $R =$ node

<u>Example</u> (MS) Simple normal crossings

$R_n = \dfrac{k[\![x_1, \dots, x_n]\!]}{x_1 x_2 \cdots x_n}$ Then we have.

$\begin{cases} n=1 & R = \text{field} \quad \text{Lech-stable} \\ n \geq 2 & R = \text{semi-stable}, \ \underline{\text{not}} \ \text{Lech-stable}. \end{cases}$

$$C_{LM}(R_3) = C_{LM}\left(\frac{k[\![x, y, z]\!]}{xyz}\right) = \frac{3}{2}$$

$C_{LM}(R_n) = ?$ $\leftarrow$ no closed formular for $n \geq 4$

<u>Theorem</u> $\boxed{\dim(R) = 2}$ (R, m) = complete normal

① (Goto – Iai – Watanabe)

R = Lech-Stable $\Leftrightarrow$ either R = regular or R = rational double point

(i.e., Gorenstein & pseudo-rational)

② (MS) R = lim-stable $\Rightarrow$ R = $\mathbb{Q}$-Gorenstein & log canonical

③ (MS) $\exists$ seminormal generalizations (under mild assumption)

<u>Theorem</u> (MS) R = essentially finite type / k char$(k) = 0$

Suppose R is normal and $\mathbb{Q}$-Gorenstein. Then

① R = lim-stable (e.g. semistable) $\Rightarrow$ R is log canonical

② R = Lech-stable and R = isolated singularity.

$\Rightarrow$ R is canonical.

Rmks:
- lim-stable/Lech-stable $\not\Rightarrow$ $\mathbb{Q}$-Gorenstein
- cannot drop isolated sing assumption in ②
- cannot expect "lc/canonical" $\Rightarrow$ "lim-stable/Lech-stable"

b/c $\underbrace{\text{Lech-stable}}_{\sup_{\sqrt{I}=\mathfrak{m}}\left\{\frac{e(I)}{d!\,\ell(R/I)}\right\}=1} \Rightarrow e(R)\leq d!$

semistable $\Rightarrow e(R)\leq (d+1)!$

Idea of proof of ①: lim-stable $\Rightarrow$ lc

assuming R = semistable and R is lc on punctured spec.

take a lc modification $\underbrace{Y \longrightarrow \mathrm{Spec}(R) =: X}$ this is the blowup of some $\mathfrak{m}$-primary ideal $I \subseteq \mathrm{Spec}(R)$

Consider $\overbrace{Bl_{I+t^m}(X\times\mathbb{A}^1)}^{\tilde{Y}} \longrightarrow X\times\mathbb{A}^1$ for m sufficiently divisible.

this is the blow up of $\underbrace{(I+t^m)^N}_{J} \subseteq R[t]$

$$\ell\left(R[t]/J^n\right) = \frac{e(J)}{(d+1)!}n^{d+1} + \frac{K_{\tilde{Y}}\cdot(-E)^d}{2d!}n^d + \text{lot}$$